End of Topic Test – Chapter 6: Confidence Intervals — Mark Scheme

Total: 48 marks

1. Confidence interval for a mean

$n = 25$, $\bar{x} = 132.4$, $s = 9.6$.

(a) M1 degrees of freedom $= n - 1 = 24$ identified. M1 critical value $t_{0.025,24} = 2.064$ used. M1 standard error $= s/\sqrt{n} = 9.6/5 = 1.92$. A1 margin of error $= 2.064 \times 1.92 = 3.96$. A1 CI $= (132.4 - 3.96,\ 132.4 + 3.96) = (128.4,\ 136.4)$ mmHg (1 d.p.). *[5]*

(b) B1 we are 95% confident the true (population) mean systolic blood pressure of patients at the clinic lies between 128.4 and 136.4 mmHg. B1 the "95% confidence" refers to the method: if many samples were taken and a CI constructed from each in the same way, about 95% of those intervals would contain the true population mean. B1 stated clearly in context (patients/clinic). *[3]*

(c) B2 (or B1+B1) the population standard deviation σ is unknown, so the sample standard deviation s is used as an estimate; this extra uncertainty (especially with a relatively small sample, $n = 25$) is accounted for by the t-distribution's heavier tails compared with the Normal z-distribution. *[2]*

2. Confidence interval for a proportion

$n = 400$, $x = 260$, $\hat{p} = 260/400 = 0.65$.

(a) M1 $\hat{p} = 0.65$. M1 SE $= \sqrt{\dfrac{0.65 \times 0.35}{400}} = 0.0238$. M1 margin $= 1.96 \times 0.0238 = 0.0467$. A1 CI $= (0.6033,\ 0.6967)$, i.e. (60.3%, 69.7%) to 1 d.p. *[4]*

(b) B1 compare the interval to 60%: the entire interval (60.3% to 69.7%) lies above 60%. B1 conclusion: the sample data *supports* the university's claim that at least 60% of students support the change. B1 caveat: the lower bound is close to 60%, so the claim is only just supported/should be treated with some caution. *[3]*

(c) B1 condition: $n\hat{p} \geq 5$ and $n(1 - \hat{p}) \geq 5$ (needed for the Normal approximation to be valid). B1 verified: $n\hat{p} = 400 \times 0.65 = 260 \geq 5$ and $n(1 - \hat{p}) = 400 \times 0.35 = 140 \geq 5$, so the condition holds. *[2]*

3. Sample size for a desired margin of error

$E \leq 0.5$, $\sigma \approx 3.2$, 95% confidence ($z = 1.96$).

(a) M1 formula $n \geq \left(\dfrac{z\sigma}{E}\right)^2$. M1 substitute: $n \geq \left(\dfrac{1.96 \times 3.2}{0.5}\right)^2 = (12.544)^2$. A1 $n \geq 157.3$. A1 round *up* to the next whole number: $n = 158$. *[4]*

(b) B1 the critical value z increases for 99% confidence (from 1.96 to 2.576). B1 since $n \propto z^2$, a larger z means a larger required sample size is needed to achieve the same margin of error at the higher confidence level. *[2]*

4. Critical values and confidence levels

(a) B1 $z_{0.025} = 1.96$ is the value such that 2.5% of the standard Normal distribution lies in each tail beyond ± 1.96, leaving 95% in the middle. B1 the confidence interval is constructed as $\bar{x} \pm 1.96 \times \mathrm{SE}$, using 1.96 as the multiplier of the standard error. B1 clear, complete explanation linking the shaded central region to the 95% confidence level. *[3]*

(b) B1 critical value for 99% confidence is $z = 2.576$ (2.58). B1 since this is larger than 1.96, the margin of error (and so the width of the interval) is greater – the 99% interval is wider than the 95% interval, all else being equal. *[2]*

5. Interpretation and misconceptions

(a) B3 (or B1 partial credit) correct non-technical interpretation, e.g. "We are 95% confident that the true population mean lies between 10.2 and 13.8" – i.e. if this method of estimation were repeated many times, about 95% of the resulting intervals would contain the true mean. *[3]*

(b) B1 the population mean is a fixed (though unknown) number, not a random variable, so for this particular interval it either does or does not contain the true mean – there is no "95% probability" attached to this one interval. B1 the 95% refers to the long-run performance of the *method* across repeated samples, not a probability statement about this specific interval. B1 clear overall explanation of the distinction. *[3]*

(c) B2 (or B1+B1) valid reason, e.g. a wider interval gives greater confidence of capturing the true value, which may be important for high-stakes decisions, even though it is less precise (there is a trade-off between confidence level and precision). *[2]*

6. Applying confidence intervals to decision making

$n = 50$, $\bar{x} = 99.1$, $s = 1.8$g. Target = 100g.

(a) M1 SE $= 1.8/\sqrt{50} = 0.2546$. M1 margin $= 1.96 \times 0.2546 = 0.499$. A1 CI $= (99.1 - 0.499,\ 99.1 + 0.499)$. A1 $= (98.60,\ 99.60)$ grams (2 d.p.). *[4]*

(b) B1 compare interval to target: the entire interval (98.60 to 99.60) lies below 100g. B1 since 100g is not contained in (and lies entirely above) the interval, this is evidence the true mean weight under the new process is below the target. B1 clear conclusion in context (the new process appears to have reduced the mean weight below 100g). *[3]*

(c) B1 criterion stated: lower bound of the 95% CI must not be less than 98g. B1 compare: the calculated lower bound is 98.60g, which is $\geq$ 98g. B1 conclusion: the criterion is met, so the process is judged acceptable by the quality manager's rule. *[3]*