Chapter 3: Modelling and Estimation — Mark Scheme

Values below use the reference data and formulae given at the start of the worksheet (body mass 80 kg, height 1.7 m, lifetime 75 years, heart rate 80 bpm, blood volume 5 litres, sprinting speed 10 m/s, walking speed 3 mph; Earth radius ≈ 6000 km; standard unit equivalences) unless stated otherwise. These are Fermi-estimation and modelling questions: there is no single "correct" numerical answer. Credit any answer that uses a sensible, clearly stated model/assumptions, a valid chain of reasoning, and reaches an order-of-magnitude-plausible result. Alternative reasonable assumptions leading to different final figures should be credited equally.

Exercise 3A: Standard form and Fermi estimation

1. Standard form.
   - (a) $7000 = 7 \times 10^3$
   - (b) $0.000004 = 4 \times 10^{-6}$
   - (c) $62\,000\,000 = 6.2 \times 10^7$
   - (d) $0.000023 = 2.3 \times 10^{-5}$
2. Correct standard form.
   - (a) $36 \times 10^3 = 3.6 \times 10^4$
   - (b) $420 \times 10^5 = 4.2 \times 10^7$
   - (c) $0.6 \times 10^3 = 6 \times 10^2$
   - (d) $0.05 \times 10^5 = 5 \times 10^3$
3. Calculations in standard form.
   - (a) $4 \times 10^6 \times 2 \times 10^5 = 8 \times 10^{11}$
   - (b) $(8 \times 10^8) \div (4 \times 10^3) = 2 \times 10^5$
   - (c) $(2.1 \times 10^6) \times (3 \times 10^3) = 6.3 \times 10^9$
   - (d) $(9 \times 10^6) \div (3 \times 10^4) = 3 \times 10^2$
   - (e) $3 \times 10^{-5} \times 2 \times 10^7 = 6 \times 10^2$
   - (f) $(8 \times 10^6) \div (2 \times 10^{-3}) = 4 \times 10^9$
4. Volume of the body.
   - (a) Model answer: split the body into simple solids, e.g. torso as a cuboid $0.5\text{m} \times 0.3\text{m} \times 0.6\text{m} \approx 0.09\,\text{m}^3$; head as a sphere of radius $0.1\,\text{m}$, $\frac{4}{3}\pi r^3 \approx 0.004\,\text{m}^3$; two arms as cylinders radius $0.05\,\text{m}$, length $0.7\,\text{m}$, $\pi r^2 h \approx 0.0055\,\text{m}^3$ each ($0.011\,\text{m}^3$ total); two legs as cylinders radius $0.08\,\text{m}$, length $0.9\,\text{m}$, $\approx 0.018\,\text{m}^3$ each ($0.036\,\text{m}^3$ total). Total $\approx 0.09+0.004+0.011+0.036 \approx 0.14\,\text{m}^3$. Accept any comparable cuboid/cylinder decomposition giving an answer of order $0.1\,\text{m}^3$ (i.e. roughly 50 000–150 000 cm^3), with dimensions stated.
   - (b) When floating, weight of water displaced = weight of body = 80 kg. Volume of water = mass ÷ density = $80\,000\,\text{g} \div 1\,\text{g/cm}^3 = 80\,000\,\text{cm}^3 = 0.08\,\text{m}^3$ (80 litres).
   - (c) The cuboid/cylinder model in (a) ($\approx 0.14\,\text{m}^3$) overestimates compared with the floating estimate ($0.08\,\text{m}^3$) – roughly 75% too big, though both give the same order of magnitude ($10^{-1}\,\text{m}^3$). The simple shapes in (a) do not hug the body closely (they include "extra" volume at corners/edges, and the body tapers where the model shapes do not), so (a) is a reasonable Fermi estimate but not accurate.

(d) Improve the model in (a) by: using more, smaller sections (e.g. tapering the limbs with frustums rather than uniform cylinders), taking more accurate personal measurements, allowing for the narrower neck/waist/ankles, and cross-checking against the floating method or known average body density (≈ 0.98–$1.05\,\mathrm{g/cm^3}$).

5. Pascal's constant.

   (a) Counting the digits, $k = 0.\underbrace{00\ldots0}_{30 \text{ zeros}}5492 = 5.492 \times 10^{-31}$.

   (b) Round to 1 significant figure for a Fermi estimate: $k \approx 5 \times 10^{-31}$.

   (c) $k^2 \approx (5 \times 10^{-31})^2 = 25 \times 10^{-62} = 2.5 \times 10^{-61}$.

6. Cape Town to Cairo.

   (a) Straight-line distance $\approx 8000\,\mathrm{km}$. Walking speed $3\,\mathrm{mph} \approx 4.8\,\mathrm{km/h}$. If walking ≈ 8 hours per day: time $\approx 8000 \div (8 \times 4.8) \approx 208$ days (≈ 7 months). (Accept continuous-walking answer of $8000 \div 4.8 \approx 1670$ hours ≈ 70 days, provided units/logic are clear; also accept any distance in the range 6500–9000 km with consistent working.)

   (b) Assumptions/simplifications: distance taken as a straight line (actual route/roads much longer and not walkable in places – deserts, rivers, borders, terrain); constant walking speed maintained all day; a fixed number of walking hours per day with time for rest, food, sleep; no allowance for hills, swamps, political/safety obstacles, or need for rest days/illness; assumes a continuous safe path exists.

7. Light years.

   (a) 1 light year $= 299\,792\,458 \times (365 \times 24 \times 3600)\,\mathrm{m} = 299\,792\,458 \times 31\,536\,000 \approx 9.46 \times 10^{15}\,\mathrm{m}$.

   (b) 8.6 light years $= 8.6 \times 9.46 \times 10^{15} \approx 8.14 \times 10^{16}\,\mathrm{m}$.

   (c) Speed $= 5.9 \times 10^4\,\mathrm{km/h} = 5.9 \times 10^7\,\mathrm{m/h}$. Distance $= 8.14 \times 10^{16}\,\mathrm{m}$. Time $= \dfrac{8.14 \times 10^{16}}{5.9 \times 10^7} \approx 1.38 \times 10^9$ hours $\approx \dfrac{1.38 \times 10^9}{24 \times 365} \approx 1.6 \times 10^5$ years (about 157 000 years).

8. Blood volume.

   (a) Using the floating-body estimate from Q4(b), $0.08\,\mathrm{m^3} = 80$ litres. 7% of 80 litres = 5.6 litres – close to the reference figure of 5 litres, which is a good check on the model. (If the cuboid estimate from Q4(a) is used instead, 7% of ≈ 140 litres ≈ 9.8 litres.)

   (b) Heart pumps the full blood volume (5 litres) every minute. Lifetime in minutes $= 75 \times 365 \times 24 \times 60 \approx 3.94 \times 10^7$ minutes. Total volume pumped $\approx 5 \times 3.94 \times 10^7 \approx 1.97 \times 10^8$ litres $\approx 2 \times 10^8$ litres (roughly 200 million litres, i.e. about $197\,000\,\mathrm{m^3}$).

9. Two further questions about Amina's water collection (any two sensible questions, e.g.): "How many trips does she make each day?"; "How much water can she carry in one container/trip?"; "How long does a round trip take, including queuing?"; "Does anyone help her carry water?"

10. Amina's water use.

    (a) Main uses: drinking, cooking, washing (people and clothes), cleaning, watering animals or crops.

    (b) Estimate using basic per-person needs, e.g. drinking $\approx 3\,\mathrm{L}$, cooking $\approx 3\,\mathrm{L}$, washing/cleaning ≈ 10–$15\,\mathrm{L}$ per person per day $\Rightarrow \approx 15$–$20\,\mathrm{L}$/person/day. For a family of 6 (Amina, husband, 4 children): ≈ 90–150 litres/day. Accept any figure in this range with reasoning shown (this matches the commonly used WHO minimum of $\sim 20\,\mathrm{L}$/person/day for basic needs).

11. Time spent collecting water. Round trip distance $= 2 \times 8 = 16\,\text{km}$. Walking with a heavy container likely slower than normal walking pace, e.g. $\approx 4\,\text{km/h} \Rightarrow$ walking time ≈ 4 hours, plus queuing/filling time (≈ 0.5–1 hour). A single trip (using, say, a 20 L container) cannot supply the whole family's daily need (Q10(b)), so in practice she may only collect enough for priority uses or make more than one trip. A sensible Fermi estimate is ≈ 3–5 hours per day. Effect on daily life: little time left for school, paid work, other household tasks, rest or childcare; physical strain and health risk from carrying heavy loads over the terrain; opportunity cost that particularly limits girls'/women's education and income.

12. Global access to safe water.

    (a) World population $\approx 8 \times 10^9$. 1 in 10 lack safe water: $\approx 8 \times 10^9 \div 10 = 8 \times 10^8$ (about 800 million people).

    (b) UK population $\approx 6.7 \times 10^7$ (67 million). 800 million $\div$ 67 million ≈ 12. So the number of people without safe water access worldwide is roughly 12 times the population of the whole UK.

13. UK domestic water use.

    (a) Domestic uses: drinking, cooking, washing dishes, bathing/showering, washing clothes, flushing the toilet, watering the garden.

    (b) Personal/family estimate: accept any reasoned figure, e.g. ≈ 140–150 litres/person/day $\Rightarrow$ a family of 4 uses roughly 500–600 litres/day.

    (c) Family of four: $\approx 4 \times 140 = 560$ litres/day (accept any answer in the range 400–800 litres/day with reasoning).

    Activity: major non-domestic water uses in the UK: agriculture/irrigation, industry and manufacturing, power-station cooling, food and drink production, construction. (Any reasonable examples credited.)

Exercise 3B: Estimation techniques: scaling, subdividing and stating assumptions

1. Map/globe scaling (method is what is assessed: measure the distance on the map with a ruler or piece of string laid along the required path, then convert using the map's stated scale). Indicative real-world figures for checking plausibility:

    (a) North to south of South America ≈ 7000–$7500\,\text{km}$.

    (b) East to west of South America at its widest ≈ 5000–$5500\,\text{km}$.

    (c) Length of the Kuroshio Current from Taiwan to Japan ≈ 1000–$2000\,\text{km}$.

    Credit any answer obtained by correctly applying a stated map scale, even if the figure differs from those above.

2. Pulse-rate estimate. Using the reference heart rate of 80 bpm (i.e. ≈ 13 beats in 10 seconds): beats per year $= 80 \times 60 \times 24 \times 365 \approx 4.2 \times 10^7$. Over 21 years: $\approx 4.2 \times 10^7 \times 21 \approx 8.8 \times 10^8$ (about 880 million beats). Accept any answer consistent with the student's own measured 10-second pulse count, scaled up correctly, and noting (for extra credit) that resting heart rate is not constant throughout childhood/adulthood so this is only an approximation.

3. Idris and Madagascar.

    (a) He may not have noticed that a globe is curved: measuring a length directly off a photo/image of a sphere is affected by foreshortening (the surface curves away from the viewer, especially away from the centre of the image), so a straight ruler measurement does not correspond to the true surface distance. (Also accept: the scale marked on a globe is only accurate at the equator/along great circles he used and distorts elsewhere, or he did not follow the curved coastline.)

(b) This would make his estimate too small (foreshortening/curvature makes the length look shorter than it really is).

(c) Improvements: use a flat map with an accurate printed scale for that region instead of a photo of a curved globe; rotate the globe so Madagascar directly faces the viewer (minimising foreshortening); lay a flexible string along the actual coastline/length on the globe and then measure the string against the scale; cross-check with atlas/internet figures.

4. Steamship speed. Southampton to New York $\approx 5500\,\text{km}$ (accept 5000–5800 km). Time = 6 days = 144 hours. Speed $= 5500 \div 144 \approx 38\,\text{km/h}$. (Accept 35–40 km/h range depending on distance assumed.)

5. Factors affecting crossing time.

   (a) Weather (storms, headwinds/tailwinds), sea state, ocean currents (e.g. Gulf Stream), icebergs/ice fields requiring slower speeds or detours, ship design/engine power/age, route chosen (shortest vs. safest), any scheduled stops.

   (b) Captains could choose routes exploiting favourable currents and avoiding storm systems/icebergs (using weather/ice reports), take the most direct safe route, keep engines well maintained, and use faster, more powerful ships.

6. Fermi estimate of the extra time for slower ships. Using Q4's average speed ($\approx 38\,\text{km/h}$) and assuming a slower ship (worse weather/currents/older engines, from Q5) averages, say, 15–20% less speed ($\approx$ 30–32 km/h) over the same $\approx 5500\,\text{km}$ route: time $\approx 5500 \div 31 \approx 177$ hours ≈ 7.4 days, i.e. roughly 1–2 days longer than the faster ships. Accept any answer in the range "half a day to a few days longer" provided a clear chain of assumptions (assumed speed reduction, same distance) is shown.

Consolidation Exercise 3

These are open Fermi/modelling questions. The indicative solutions below show a valid modelling chain (assumptions → calculation → order-of-magnitude answer); credit equivalent reasoning with different but reasonable assumptions.

Human beings

1. Blinks in a lifetime: assume ≈ 15 blinks/minute while awake, awake ≈ 16 hours/day. $15 \times 60 \times 16 \times 365 \times 75 \approx 3.9 \times 10^8$ (about 400 million blinks).

2. Time spent queueing in a lifetime: assume ≈ 10 minutes/day spent queueing (shops, traffic, etc.). $10 \times 365 \times 75 = 273\,750$ minutes ≈ 4560 hours ≈ 190 days (roughly half a year). Accept 5–20 minutes/day assumptions giving answers from a couple of months to about a year.

3. Distance a person can kick a football: a powerful, well-struck kick by a fit adult travels very roughly half the length of a football pitch; estimate $\approx$ 40–50 m (accept 20–70 m with a stated basis, e.g. comparison with a goal-kick or known pitch length of 100 m).

4. Breaths in a lifetime: assume ≈ 15 breaths/minute, continuously (day and night). $15 \times 60 \times 24 \times 365 \times 75 \approx 5.9 \times 10^8$ (about 600 million breaths).

Meteorology

5. Volume of snow falling on Scotland each year: area of Scotland $\approx 80\,000\,\text{km}^2 = 8 \times 10^{10}\,\text{m}^2$. Assume an average annual fallen-snow depth (accumulated depth of snow as it falls, not water equivalent), spread unevenly with much more in the highlands and little at low levels, averaging

perhaps 0.3–0.5 m over the whole area. Volume $\approx 8 \times 10^{10} \times 0.4 \approx 3 \times 10^{10}\,\text{m}^3$. Accept any answer of order 10^9–$10^{11}\,\text{m}^3$ with area and depth assumptions stated.

Food and drink

6. Weight of bread consumed in the UK per year: UK population $\approx 6.7 \times 10^7$. Assume ≈ 1 (800 g) loaf/person/week $\Rightarrow\approx 41.6$ kg/person/year. Total $\approx 6.7 \times 10^7 \times 41.6 \approx 2.8 \times 10^9$ kg ≈ 2.8 million tonnes/year (accept 1–4 million tonnes).
7. Fraction of an Olympic pool filled by lifetime tea/coffee: pool volume $= 50 \times 25 \times 2 = 2500\,\text{m}^3 = 2.5 \times 10^6$ litres. Assume 3 cups/day at 250 ml $= 0.75$ L/day. Lifetime volume $= 0.75 \times 365 \times 75 \approx 2.05 \times 10^4$ litres. Fraction $\approx \dfrac{2.05 \times 10^4}{2.5 \times 10^6} \approx 0.008$, i.e. under 1% of the pool (roughly 1/120).
8. Cups of tea drunk per day in the UK: population $\approx 6.7 \times 10^7$, assume an average (including non-drinkers) of ≈ 2–3 cups/person/day $\Rightarrow\approx 1.3$–2×10^8 cups/day (100–200 million cups/day). Accept any figure of order 10^8.

Science

9. Mass of the Antarctic ice sheet: volume $= 2.6 \times 10^7\,\text{km}^3 = 2.6 \times 10^{22}\,\text{cm}^3$ (since $1\,\text{km}^3 = 10^{15}\,\text{cm}^3$). Mass $= 0.92 \times 2.6 \times 10^{22} \approx 2.4 \times 10^{22}\,\text{g} = 2.4 \times 10^{19}$ kg ($\approx 2.4 \times 10^{16}$ tonnes).
10. Speed of Nairobi due to Earth's rotation: Nairobi is very close to the equator, so radius of rotation ≈ 6000 km. Circumference $= 2\pi \times 6\,000\,000 \approx 3.77 \times 10^7$ m. Time $= 1$ day $= 86\,400$ s. Speed $\approx 3.77 \times 10^7 \div 86\,400 \approx 436$ m/s ≈ 1570 km/h.
11. Speed of Nairobi orbiting the Sun: this equals the Earth's orbital speed. Using Earth–Sun distance $\approx 1.5 \times 10^8\,\text{km} = 1.5 \times 10^{11}$ m (a value that must be recalled/looked up, as it is not in the given reference data): orbit circumference $= 2\pi \times 1.5 \times 10^{11} \approx 9.42 \times 10^{11}$ m; time $= 1$ year $\approx 3.156 \times 10^7$ s. Speed $\approx 9.42 \times 10^{11} \div 3.156 \times 10^7 \approx 2.98 \times 10^4$ m/s ≈ 29.8 km/s ($\approx 107\,000$ km/h).

Economics

12. Weight of recycling collected annually in the UK: ≈ 27 million households, assume ≈ 5 kg/week each. $27 \times 10^6 \times 5 \times 52 \approx 7 \times 10^9$ kg ≈ 7 million tonnes/year (accept 3–12 million tonnes).
13. Lifetime spend on food: assume $\approx$ £60/week on food. $60 \times 52 \times 75 =$ £234 000. Accept any figure in the range £150 000–£300 000 depending on the weekly-spend assumption stated.
14. Annual electricity cost for UK homes: ≈ 28 million households, average bill $\approx$ £800/year. Total $\approx 28 \times 10^6 \times 800 \approx$ £2.2×10^{10} ($\approx$ £22 billion/year). Accept £15–25 billion with reasoning.

Technology

15. Blu-ray pit spacing: assume single-layer capacity 25 GB $= 25 \times 10^9$ bytes $= 2 \times 10^{11}$ bits, stored on the data zone of a 12 cm disc between radii ≈ 24 mm and ≈ 58 mm. Area $= \pi(58^2 - 24^2)\,\text{mm}^2 \approx 8760\,\text{mm}^2 = 8.76 \times 10^9\,\mu\text{m}^2$. For a square array, spacing d satisfies $d^2 =$ area $\div$ number of bits $= 8.76 \times 10^9 \div 2 \times 10^{11} \approx 0.044\,\mu\text{m}^2$, so $d \approx 0.21\,\mu\text{m} \approx 210$ nm. This is the right order of magnitude for real Blu-ray pit/track spacing ($\sim$100–300 nm), which is what the question expects.

Miscellany

16. People using a mobile phone at any one time worldwide: assume $\approx 5 \times 10^9$ people own a phone, and that (allowing for time zones, sleep, etc.) ≈ 10–15% are actively using one at any given moment $\Rightarrow\approx 5$–7.5×10^8 (roughly 500–750 million people). Accept any order-10^8–10^9 answer with reasoning.

17. Roof tiles in the UK: $\approx$ 28 million homes, average roof area (both sides) $\approx 80\,\text{m}^2$, tile $\approx 40\times20$ cm with overlap giving effective coverage $\approx 0.04\,\text{m}^2/\text{tile} \Rightarrow\approx 2000$ tiles/roof. Total $\approx 28\times10^6\times 2000 = 5.6\times10^{10}$ ($\approx$ 56 billion tiles). Accept order 10^{10}–10^{11}.

18. Weight of fallen leaves collected each year from a typical UK garden: assume a garden with a few trees producing a total of ≈ 50 kg of leaves in autumn. Accept any figure in the range 20–150 kg with reasoning.

Investigation: The Greenland Ice Sheet

Total amount of ice. Area $= 2\,170\,000\,\text{km}^2$; depth varies 0–3000 m. A simple Fermi model treats the ice sheet as a dome and takes the *average* depth as roughly half the maximum depth, ≈ 1500 m $= 1.5$ km (state this assumption clearly – a more sophisticated model might use a higher fraction, e.g. $\frac{2}{3}$, since a dome profile is thicker than a cone over most of its area).

$$\text{Volume} \approx 2\,170\,000\,\text{km}^2 \times 1.5\,\text{km} \approx 3.3\times10^6\,\text{km}^3 = 3.3\times10^{15}\,\text{m}^3$$

Using ice density $\approx 0.92\,\text{g/cm}^3 = 920\,\text{kg/m}^3$:

$$\text{Mass} \approx 920\times3.3\times10^{15} \approx 3.0\times10^{18}\,\text{kg}$$

Effect on sea level. If all this ice melted (as land ice, this adds new water to the oceans – unlike floating sea ice), volume of meltwater = mass $\div$ density of water $= 3.0\times10^{18}\div1000 \approx 3.0\times10^{15}\,\text{m}^3$. Taking the ocean's surface area as $\approx 3.6\times10^8\,\text{km}^2 = 3.6\times10^{14}\,\text{m}^2$:

$$\text{Sea level rise} \approx \frac{3.0\times10^{15}}{3.6\times10^{14}} \approx 8\,\text{m}$$

(This is a good order-of-magnitude match to the widely quoted real figure of ≈ 7.4 m, which is reassuring given the simplicity of the average-depth assumption.) Accept any answer in the range roughly 5–10 m obtained from a clearly stated average-depth assumption.

Critical evaluation (indicative points, credit any well-argued selection):

- The average-depth assumption is crude – real ice thickness is not a simple linear taper from centre to edge, so "half the maximum depth" may under- or over-estimate the true average.
- Ice density increases somewhat with depth/pressure near the base, so using a single density slightly underestimates mass.
- Ocean surface area was treated as fixed, but as sea level rises the ocean encroaches on low-lying land, slightly increasing the area over which the extra water spreads (making the true rise a little less than calculated) – and land also rises via isostatic rebound once ice is removed, an effect ignored here.
- Not all meltwater reaches the ocean immediately (some may be absorbed, refrozen elsewhere, or affect ocean circulation/temperature), and thermal expansion of warming seawater is a separate contributor to sea-level rise not included in this model.
- The model is static (a one-off calculation) and says nothing about the time scale of melting, whereas the real process is gradual and its rate is what is actually observed (and reported) to be accelerating.

Significance of conclusions: Even a simple, approximate model shows that complete melting of the Greenland ice sheet would raise global sea levels by several metres. Since a very large number of the world's major cities and much productive land lie within a few metres of current sea level, this represents a potentially catastrophic and largely irreversible impact, underlining why the observed long-term mass loss (even though individual glaciers show short-term local variation) is considered a serious consequence of climate change rather than a matter for concern only in the far future.

Review

This is a self-assessment checklist for students (standard form and calculation with it; Fermi estimation and stating assumptions; scaling and subdividing; using simple formulae for volume, area, speed and density; critically evaluating a model against real data) and does not require a mark-scheme answer. Teachers may wish to use the Investigation and Consolidation Exercise questions above as evidence towards these outcomes.