Chapter 6: Confidence Intervals — Mark Scheme

Note: the source worksheet (Exercises 6A–6C) covers point estimates, the distribution of the sample mean, and confidence intervals, in line with the "Confidence Intervals" heading printed on the worksheet itself.

Exercise 6A — Point estimates

1. Point estimate $= \dfrac{452 + 447 + 455 + 449 + 450 + 453}{6} = \dfrac{2706}{6} = 451$ g.
2. The second officer's estimate (448.9 g) is likely to be more accurate, since it is based on a much larger sample (30 jars rather than 6), so it is less affected by chance variation between individual jars.
3. Combining both samples: total mass $= 2706 + 30 \times 448.9 = 2706 + 13467 = 16173$ g from $6 + 30 = 36$ jars.
$$\text{Point estimate} = \frac{16173}{36} = 449.25 \text{ g.}$$
4. $X \sim N(500, 4^2)$. $z = \dfrac{495 - 500}{4} = -1.25$.
$$P(X < 495) = \Phi(-1.25) = 1 - \Phi(1.25) = 1 - 0.8944 = 0.1056.$$

Exercise 6B — The distribution of the sample mean

1. $n = 25$, $X \sim N(50, 10^2)$, so standard error $= \dfrac{10}{\sqrt{25}} = 2$, i.e. $\bar{X} \sim N(50, 2^2)$.
   (a) $z = \dfrac{52 - 50}{2} = 1$, so $P(\bar{X} > 52) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587$.
   (b) $z = \dfrac{47 - 50}{2} = -1.5$, so $P(\bar{X} < 47) = \Phi(-1.5) = 1 - 0.9332 = 0.0668$.
   (c) $z_1 = \dfrac{48 - 50}{2} = -1$, $z_2 = \dfrac{53 - 50}{2} = 1.5$.
$$P(48 < \bar{X} < 53) = \Phi(1.5) - \Phi(-1) = 0.9332 - 0.1587 = 0.7745.$$
2. $X \sim N(62, 5^2)$, $n = 16$: standard error $= \dfrac{5}{\sqrt{16}} = 1.25$. $z = \dfrac{60 - 62}{1.25} = -1.6$, so
$$P(\bar{X} < 60) = \Phi(-1.6) = 1 - \Phi(1.6) = 1 - 0.9452 = 0.0548.$$
3. $X \sim N(35, 8^2)$, $n = 20$: standard error $= \dfrac{8}{\sqrt{20}} = 1.789$. $z = \dfrac{37 - 35}{1.789} = 1.118$, so $P(\bar{X} > 37) = 1 - \Phi(1.118) \approx 1 - 0.8682 = 0.1318$. Expected number of samples $= 100 \times 0.1318 \approx 13$ samples.
4. $X \sim N(120, 45)$, $n = 9$: standard error $= \sqrt{\dfrac{45}{9}} = \sqrt{5} = 2.236$.
   (a) $z = \dfrac{123 - 120}{2.236} = 1.342$, so $P(\bar{X} > 123) = 1 - \Phi(1.342) \approx 1 - 0.9101 = 0.0899$.
   (b) $z_1 = \dfrac{117 - 120}{2.236} = -1.342$, $z_2 = \dfrac{124 - 120}{2.236} = 1.789$.
$$P(117 < \bar{X} < 124) = \Phi(1.789) - \Phi(-1.342) \approx 0.9633 - 0.0899 = 0.8734.$$

5. $P(11.5 < V < 12.5) = 0.68268$ is precisely $P(-1 < Z < 1)$, so the interval $(11.5, 12.5)$ corresponds to $\bar{\mu} \pm 1 \times SE$, i.e. the standard error is $12.5 - 12 = 0.5$.

$$SE = \frac{\sigma}{\sqrt{n}} = \frac{2}{\sqrt{n}} = 0.5 \;\Rightarrow\; \sqrt{n} = 4 \;\Rightarrow\; n = 16,$$

as required.

Exercise 6C — Confidence intervals

1. $n = 50$, $\sigma = 120$, $\bar{x} = 1400$: standard error $= \dfrac{120}{\sqrt{50}} = 16.97$.

$$90\%\text{ CI: } 1400 \pm 1.64 \times 16.97 = 1400 \pm 27.83 = (1372.2,\ 1427.8)\text{ hours (1 d.p.).}$$

2. $n = 9$, $\sigma = 2.4$, $\bar{x} = 34.8$: standard error $= \dfrac{2.4}{\sqrt{9}} = 0.8$.

$$99\%\text{ CI: } 34.8 \pm 2.58 \times 0.8 = 34.8 \pm 2.064 = (32.7,\ 36.9)\text{ g (1 d.p.).}$$

3. $n = 25$, $\sigma = 60$, $\bar{x} = 2978$: standard error $= \dfrac{60}{\sqrt{25}} = 12$.

$$95\%\text{ CI: } 2978 \pm 1.96 \times 12 = 2978 \pm 23.52 = (2954.5,\ 3001.5)\text{ mAh (1 d.p.).}$$

The claimed mean of 3000 mAh lies *inside* this interval, so on the basis of this sample there is no reason to doubt the manufacturer's claim.

4. A 99% confidence interval would be **wider** than a 95% one. A higher confidence level requires a larger z-value (2.58 instead of 1.96), which increases the margin of error $z \times \dfrac{\sigma}{\sqrt{n}}$, so the interval must be wider to be more confident it captures the true mean.

5. **Increase the sample size n.** Since the standard error is $\dfrac{\sigma}{\sqrt{n}}$, increasing n decreases the standard error, which decreases the margin of error $z \times \dfrac{\sigma}{\sqrt{n}}$ and so narrows the confidence interval, without changing the confidence level.